41st **Balkan Mathematical Olympiad**
27th April – 2nd May 2024
Varna, Bulgaria

Language: English

Monday, April 29, 2024

Problem 1. Let ABC be an acute-angled triangle with $AC > AB$ and let D be the foot of the A-angle bisector on BC. The reflections of lines AB and AC in line BC meet AC and AB at points E and F respectively. A line through D meets AC and AB at G and H respectively such that G lies strictly between A and C while H lies strictly between B and F. Prove that the circumcircles of $\triangle EDG$ and $\triangle FDH$ are tangent to each other.

Problem 2. Let $n \geq k \geq 3$ be integers. Show that for every integer sequence $1 \leq a_1 < a_2 < \ldots < a_k \leq n$ one can choose non-negative integers $b_1, b_2, \ldots, b_k$, satisfying the following conditions:

(i) $0 \leq b_i \leq n$ for each $1 \leq i \leq k$,

(ii) all the positive b_i are distinct,

(iii) the sums $a_i + b_i$, $1 \leq i \leq k$, form a permutation of the first k terms of a non-constant arithmetic progression.

Problem 3. Let a and b be distinct positive integers such that $3^a + 2$ is divisible by $3^b + 2$. Prove that $a > b^2$.

Problem 4. Let $\mathbb{R}^+ = (0, \infty)$ be the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ and polynomials $P(x)$ with non-negative real coefficients such that $P(0) = 0$ which satisfy the equality

$$f(f(x) + P(y)) = f(x - y) + 2y$$

for all real numbers $x > y > 0$.

Time is 4 hours and 30 minutes
Each problem is worth 10 points